USING DIRECT METHODS TO SOLVE COUPLED CONSOLIDATION EQUATIONS

Felicja Okulicka-Dłużewska
Faculty of Mathematics and Information Sciences
Warsaw University of Technology
00-661 Warszawa
Plac Politechniki 1
Poland
okulicka@mini.pw.edu.pl

Description of different mechanical phenomena such as flow, mechanical behaviour, thermal effects, leads to coupled systems of differential equations. To solve a certain initial boundary value problems, the finite element methods can be used. In such situation where a few phenomena are taken into account the final form of global equation set takes the block form. In general, the times when important phenomena has been considered separately belongs to the past. Now we want to model very complicated and complex effects. For example if we consider the car engine we have to solve mechanical and thermal differential equations as a coupled system. The coupled systems appear in modern mechanics very often. Ground waters flow and mechanical behaviour deformation and stressed, transport of the pollutants, thermal flow etc. Coupled problems are much more complicated, comparing each effect considered separately but solving of them gives very realistic behaviour of complex problems.

If we consider the thermal flow in the saturated soil, the system of the differential equations describing the behaviour of such medium is coupled with the mechanical part. Applying the finite element method to solve initial boundary vale problems we arrive at the coupled block system of global equation set which have to be solved. The solution of the system is one of the most time consuming procedures of the whole calculation process. The coupled global system of equations takes the following form [11]

$$\begin{bmatrix} \mathbf{K} & \mathbf{L1} & \cdots & \mathbf{W1} \\ \mathbf{L2} & \mathbf{S} & \cdots & \mathbf{R1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{W2} & \mathbf{R2} & \cdots & \mathbf{V} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}}{dt} \\ \frac{d\mathbf{p}}{dt} \\ \vdots \\ \frac{d\mathbf{T}}{dt} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \cdots & \mathbf{C} \\ \mathbf{B} & \mathbf{D} & \cdots & \mathbf{E} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C} & \mathbf{E} & \cdots & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \vdots \\ \mathbf{t} \end{bmatrix} + \begin{bmatrix} \frac{d\mathbf{P}}{dt} \\ \mathbf{Q} \\ \vdots \\ \mathbf{Y} \end{bmatrix} \tag{1}$$

In the above system each equation in row describes different effect. Each new effect taken into account enlarges the global equation (1) by the new block. When the finite differences method is used for the solution of the initial value problem [7] the time stepping process became highly time consuming.

As an example, the soil consolidation problem in terms of the finite element method is discussed. The detail definitions of the arrays can be found in the book [5] and papers [4,6]. The FEM model applied in standard way leads to the band matrix equations. In the paper the block equations are analyzed comparing with the band form. The nodal equilibrium equation is coupled with the mass balance equation via volumetric changes. Both equations can be linked in the following block form

$$\begin{bmatrix} \mathbf{K}_T & \mathbf{L} \\ \mathbf{L}^T & -\mathbf{S} \end{bmatrix} \begin{bmatrix} \frac{d\mathbf{u}}{dt} \\ \frac{d\mathbf{p}}{dt} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} + \begin{bmatrix} \frac{d\mathbf{P}}{dt} \\ \mathbf{Q} \end{bmatrix} \tag{2}$$

The following definitions are applied: **H** - array responsible for flow properties, **S** - array responsible for water compressibility, **L** - coupling array, **Q** - nodal discharges vector, **P** - vector of load, **u** - vector of displacement, **p** - vector of the excess pore-water pressure.

The above set of equations describes the behaviour of the two-phase medium in terms of the finite element approximation.

Matrixes K and S

- modelled different effects
- have different range of elements
- can be generated concurrently (there is no need to send matrixes from one process to another)

For a 2-node element the field variables are assumed to vary lineary between the nodal values. The various sub-matrices in a coefficient matrix can be evaluated as:

$$\mathbf{K} = \frac{\mathbf{aD}}{\mathbf{h}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$
$$\mathbf{L} = \mathbf{a}\begin{bmatrix} \mathbf{-0,5} & \mathbf{-0,5} \\ \mathbf{0,5} & \mathbf{0,5} \end{bmatrix} \qquad (3)$$
$$\mathbf{S} = \frac{\mathbf{ak\Delta t}}{\gamma_W \mathbf{h}}\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

where

D=[E’(1-ν’)]/[(1+ν’)(1-2ν’)]
E’ – effective stress Young’s modulus
ν’ – effective stress Poisson’s ratio
a – cross section area of the element
h – element size
k – permeability matrix
Δt – time step
γ_W – unit weight of a water

The values of the elements of the matrixes depend on the coefficients k and E and the mesh. The effective stress Young’s modulus E’ is on the range > 10^3, the permeability matrix mulyiplied bythe time step Δt is on the range 10^{-3}.

The block formulation of the coupled problems makes natural the application of the block methods for solving the sets of linear equations. The large matrixes can be split into blocks and put into separate memories of the net of computers. The parallel calculations are reached due to the matrix operations on separate blocks. The standard numerical algorithms should be rebuilt for the block version.

<u>Theorem</u>

The error in the block algorithm with 4 blocks (2) is n times less than in the case with band matrix algorithm, where:

n= $O(D\gamma_W/||k||\Delta t)$ (> 10^6)

When we divide matrix on more blocks, the error grows, but accurancy of the solution is still better then in the case where we solve the band matrix form of the same equations.

The accurancy is tested for different matrix division on blocks. The theorethical calculation is made. The comparison on accurancy for different division on blocks is presented.

The parallel algorithms to solve coupled consolidation problem is implementing and testing in the package HYDRO-GEO.

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