Proving that a set is pathwise-connected with interval arithmetic

Nicolas Delanoue, Luc Jaulin, Bertrand Cottenceau

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Abstract : In this paper, we give a numerical algorithm able to prove whether a set $\mathbb{S}$ described by nonlinear inequalities is pathwise-connected or not. To our knowledge, no other algorithm (numerical or symbolic) is able to deal with this type of problem in a reliable way, except for academic examples. The proposed approach uses interval arithmetic to build a graph which has exactly the same number of connected components than $\mathbb{S}$. An illustrative example shows the principle of the approach.

Introduction

Topology is the mathematical study of properties of objects which are preserved through deformations, twistings, and stretchings. Because spaces by themselves are very complicated, they are unmanageable without looking at particular aspects. One of the topological aspects of a set is its number of pathwise-connected components.

Proving that a set is connected is an important problem already consider for robotics and identifiability applications ([11], [8]).

In section 1, some notions of topology are recalled. The next section deals with lattices and intervals. Most of the examples presented in this section are useful to understand the proposed reliable method. The section 3 shows how a specific problem of topology can be solved by resolving a constraint satisfaction problem [7]. In the last part, the method is given with an illustrated example.

1 Topology recall

Definition 1.1 (***pathwise-connected set***)
A topological space X is *pathwise-connected* if and only if for every two points $x, y \in X$, there is a continuous function f from $[0, 1]$ to X such that $f(0) = x$ and $f(1) = y$. Pathwise-connected spaces are also called 0-connected. (For more about topology, see [3] and [5]).

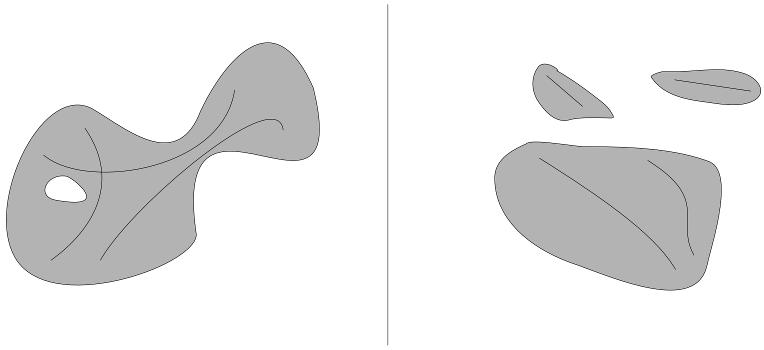

Figure 1: Example of a pathwise-connected set(left) and a non pathwise-connected set (right) in $\mathbb{R}^2$

Definition 1.2 (*Star*)
The point v^* is a *star* for a subset X of an Euclidean space if X contains all the line segment connecting any of its points and v^*.

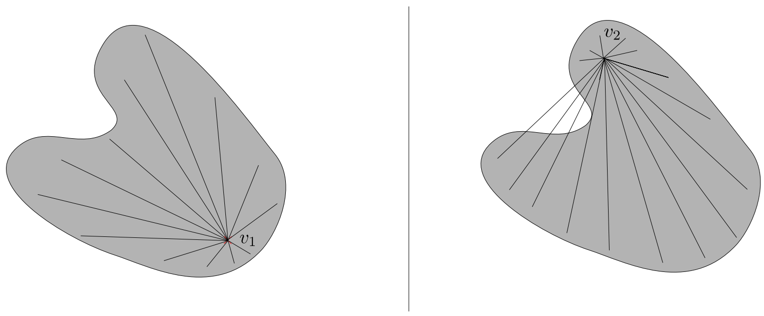

Figure 2: v_1 is a star for this subset of $\mathbb{R}^2$ and v_2 is not.

Definition 1.3 (*Starry*)
If there exists $v^* \in X$ such that v^* is a star for X, then we say that X is *starry* or v^**-starry*.

Proposition 1.1 *A starry set is a pathwise-connected set.*

Proposition 1.2 *Let X and Y two v^*-starry set, then $X \cap Y$ is v^*-starry.*

Remark 1.1 If X is convex then all $x \in X$ is a star for X.

2 Interval arithmetic in lattices

This section recalls some definitions and properties related to lattices.

Definition 2.1
A lattice $(X, \leq)$ is a partially ordered set satisfying :

$$\forall x, y \in X, x \vee y \in X \text{ and } x \wedge y \in X,$$

where $x \wedge y$ is the greatest lower bound and is called the *meet*, $x \vee y$ is the least upper bound and is called the *join*. See [1] and [2] for more details.

Example 2.1 *The set* $\mathbb{R}$ *with the total order* $\leq$ *is a lattice, in this case*

$$x \vee y = max(x, y) \text{ and } x \wedge y = min(x, y).$$

Example 2.2 *Using total order* $\leq$ *on* $\mathbb{R}$*, it is possible to make* $\mathbb{R}^2$ *become naturally a lattice : Let* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$ *in* $\mathbb{R}^2$*,*

$$x \leq_2 y \Leftrightarrow \begin{cases} x_1 \leq y_1 \\ x_2 \leq y_2 \end{cases}.$$

According to this partial order relation, the lattice operations are defined by $x \vee_2 y = (x_1 \vee y_1, x_2 \vee y_2)$ *and* $x \wedge_2 y = (x_1 \wedge y_1, x_2 \wedge y_2)$*.*

Example 2.3 *But this is not the only way to make* $\mathbb{R}^2$ *become a lattice, we could create the partial relation* $\leq_3$

$$x \leq_3 y \Leftrightarrow \begin{cases} x_1 + x_2 \leq y_1 + y_2 \\ x_1 - x_2 \leq y_1 - y_2 \end{cases}.$$

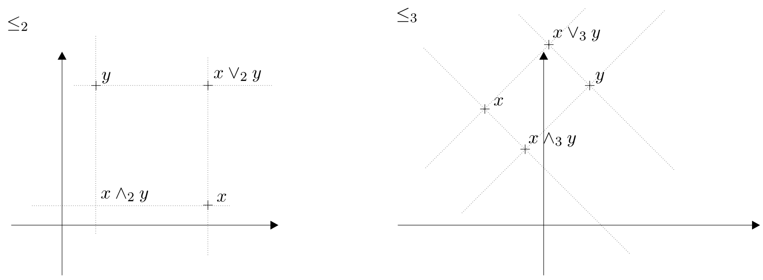

Figure 3: Some order relations on $\mathbb{R}^2$

Example 2.4 *With* $\mathcal{P}(\mathbb{R}^2)$ *the set of all subsets of* $\mathbb{R}^2$*,* $(\mathcal{P}(\mathbb{R}^2), \subset)$ *is a lattice. In this case* $X \vee Y = X \cup Y$ *and* $X \wedge Y = X \cap Y$*.*

Example 2.5 *Let* E *be a set, a simple graph on* E *is a symmetric relation on* E*, i.e. a subset of* $E \times E$*. Let* G *be the set of all simple graphs on* E*,* G *is a lattice with respect to the partial order :* $g_1, g_2 \in G$*. (See [4])*

$$g_1 \leq g_2 \Leftrightarrow g_1 \subset g_2.$$

When E *is finite, a simple graph can be represented by nodes and edges. The next figure uses this representation to illustrate the order in* G*.*

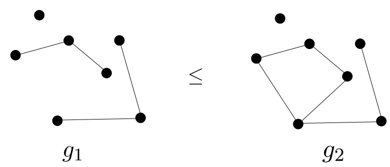

Figure 4: Example $g_1 \leq g_2$

Definition 2.2 (Interval)
An *interval* I of a lattice ξ is a subset of ξ which satisfies

$$I = \{x \in \xi \text{ such that } \wedge I \leq x \leq \vee I\}$$

Note that here, $\vee I$ and $\wedge I$ belong to $\overline{\xi}$, but may not belong to ξ. Both $\emptyset$ and ξ are intervals of ξ. The set of all intervals of ξ will be denoted $\mathcal{I}(\xi)$. Note that $\mathcal{I}(\xi)$ is a subset of $\mathcal{P}(\xi)$. More explanations about intervals are given in [6].

Example 2.6 $[-\pi, e^{\sqrt{2}}]$ *is an interval of* $(\mathbb{R}, \leq)$.

Example 2.7 $[(1,1),(3,4)]$ *is an interval of* $(\mathbb{R}^2, \leq_2)$ *and can be seen as the Cartesian product of the two intervals of* $(\mathbb{R}, \leq)$. *i.e.*

$$[(1,1),(3,4)] = [1,3] \times [1,4]$$

Example 2.8 $[(1,1),(3,4)]$ *is also an interval of* $(\mathbb{R}^2, \leq_3)$*, different from the last one. (See Fig* **??***).*

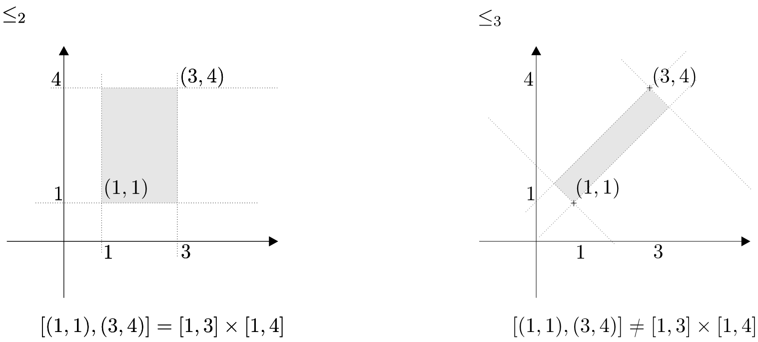

Figure 5: Intervals and relations on $\mathbb{R}^2$

Example 2.9 *With the Figure* **??***,* $[g_1, g_2]$ *is an interval of* $(\mathcal{G}, \leq)$*, this interval contains 4 elements :*

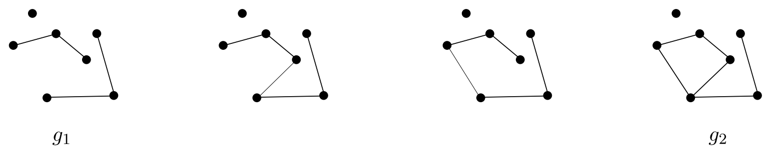

Figure 6: Example of a interval graph

This interval also can be represented as on Figure **??** *:*

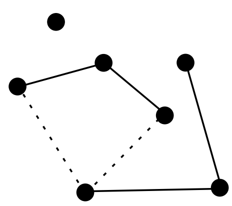

Figure 7: Representation of graph interval $[g_1, g_2]$

The edges represented by the broken lines are called undetermined edges.

3 Proving that v^* is a star

This section shows that when $\mathbb{S}$ is defined by an inequality, proving that $\mathbb{S}$ is v^*-starry amounts to proving that a set defined by inequalities is inconsistent. This result is really attractive because we will be able to use computer to show that v^* is star for $\mathbb{S}$ via an interval method. ([9], [7])

Proposition 3.1 *[To prove that v^* is a star].*
Define $\mathbb{S} = \{x \in \mathbb{R}^n$ such that $f(x) \leq 0\}$ where f is a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}$.
Let v^* be in $\mathbb{S}$. If

$$\forall x \in \mathbb{R}^n \text{ satisfying } f(x) = 0, \text{ we have } Df(x).(x - v^*) > 0 \tag{1}$$

then v^* is star for $\mathbb{S}$.

Proof 3.1 Let us note $[\alpha, \beta]$ the segment of $\mathbb{R}^n$ with endpoints α and β. and let us note by $f_{|[\alpha,\beta]}$, the function :

$$t \in [0,1] \mapsto f_{|[\alpha,\beta]}(t) = f((1-t)\alpha + t\beta) \in \mathbb{R}$$

The proof is by contradiction. Suppose that v^* is not a star for $\mathbb{S}$. Then there exists $x_0 \in \mathbb{S}$ such that $[v^*, x_0] \not\subset \mathbb{S}$, i.e., there exists $x_1 \in [v^*, x_0]$ such that $f(x_1) > 0$. Since the numeric function f is a $\mathcal{C}^1$ function, $f_{|[v^*,x_0]}$ is differentiable.

Moreover, it satisfies the following inequalities :

$$f_{|[v^*,x_0]}(0) \leq 0, \quad f_{|[v^*,x_0]}(1) \leq 0, \quad f_{|[v^*,x_0]}(t_1) > 0.$$

where t_1 is such that $x_1 = (1-t_1)v^* + (t_1)x_0$.

Since $f_{|[v^*,x_0]}$ is continuous, the intermediate value theorem guarantees that there exists $t_2 \in [t_1, 1]$ such that $f_{|[v^*,x_0]}(t_2) = 0$. In the case where there is more than one real in $[t_1, 1]$ which satisfies $f(x) = 0$, let t_2 be the minimum of them.
Thus we have :

$$f_{|[v^*,x_0]}(t_2) = 0 \text{ and } \forall t \in]t_1, t_2[, \quad f_{|[x_0,v^*]}(t) > 0.$$

Since $f_{|[v^*,x_0]}$ is differentiable on the open interval]0,1[,

$$f'_{|[v^*,x_0]}(t_2) = \lim_{h \to 0} \frac{f_{|[v^*,x_0]}(t_2 + h) - f_{|[v^*,x_0]}(t_2)}{h}. \tag{2}$$

$$f'_{|[v^*,x_0]}(t_2) = \lim_{h \to 0^-} \frac{f_{|[v^*,x_0]}(t_2 + h)}{h}. \tag{3}$$

There exists $\epsilon > 0$ such that $\forall h < 0, |h| < \epsilon \Rightarrow t_2 + h \in [t_1, t_2]$ (take $\epsilon = (t_1 - t_2)/2$). So

$$\forall h < 0, |h| < \epsilon, \frac{f_{|[v^*,x_0]}(t_2 + h)}{h} < 0.$$

We deduce that $f'_{|[v^*,x_0]}(t_2) \leq 0$.

In conclusion, taking $x_2 = (1-t_2)v^* + t_2 x_0$, $x_2 \in \mathbb{R}^n$ is such that :

$$f(x_2) = 0 \text{ and } Df(x_2).(x_2 - v^*) \leq 0.$$

Example 3.1 *Consider the problem of proving that* $v_1 = (0, 0.7)$ *is a star for the subset* $\mathbb{S}$ *of* $\mathbb{R}^2$ *defined by* $f(x_1, x_2) \leq 0$ *where* f *is the* C^1 *function from* $\mathbb{R}^2$ *to* $\mathbb{R}$ *defined by :*

$$f(x_1, x_2) = -e^{-(2x_1)^2} - e^{-(2x_1-2.8)^2} + 0.1 + x_2^2. \tag{4}$$

Using the proposition 3.1, this amounts to proving that the subset of $\mathbb{R}^2$ defined by

$$\begin{cases} \partial_1 f(x_1, x_2).(x_1 - 0) + \partial_2 f(x_1, x_2).(x_2 - 0.7) \leq 0 \\ f(x_1, x_2) = 0 \end{cases} \tag{5}$$

is empty.

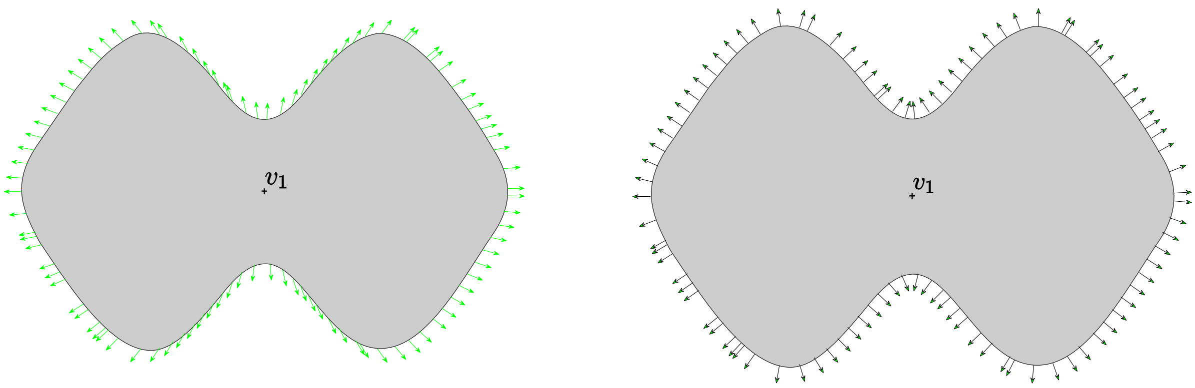

Figure 8: Left : Field unit vector $(g_i)_i$ which represents $\nabla f(x)$ where $f(x) = 0$. Right : Field unit vector $(c_i)_i$ which represents $x - v_1$ where $f(x) = 0$.

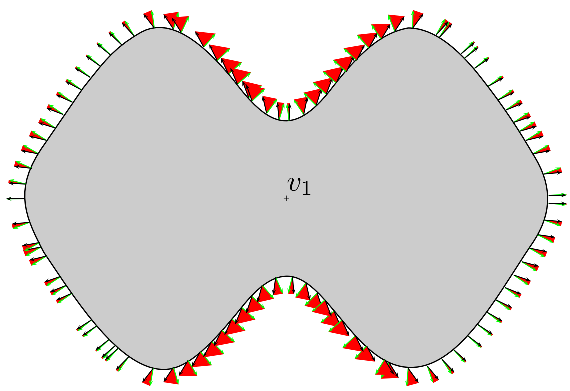

Figure 9: This figure illustrates that for all x satisfying $f(x) = 0$, we have $\nabla f(x).(v_1 - x) > 0$, i.e. the angle between two vectors c_i and g_i is an acute angle. Using our proposition, we could deduce that $\mathbb{S}$ is v_1-starry.

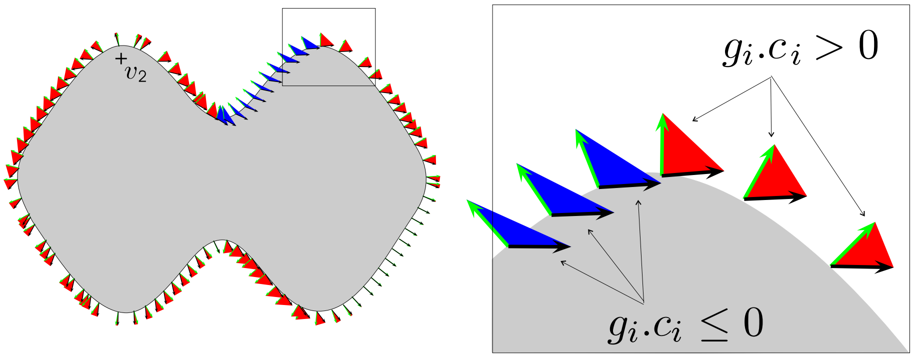

Figure 10: In this case, v_2 is not a star for X and there exists a couple (g_i, c_i) such that $g_1.c_i \leq 0$.

4 Discretization

Definition 4.1 (Paving)
Let X_0 be a subset of $\mathbb{R}^n$. A *paving* of X_0 is a set of nonoverlapping boxes such that their union is equal to X_0. On Figure **??**, a paving $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}, p_{14}\}$ of a box X_0 is represented.

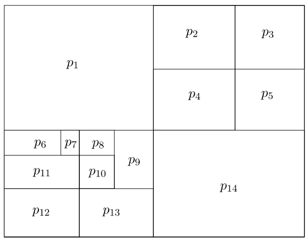

Figure 11: A paving with 14 boxes.

Recall that if a set $\mathbb{S}$ is starry then $\mathbb{S}$ is pathwise-connected (Proposition 1.1), but most of the pathwise-connected sets are not starry, the Figure **??** illustrates this case.

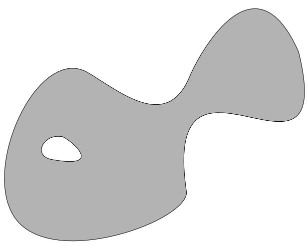

Figure 12: Example of a pathwise-connected set which is not starry.

But, often, it is possible to create a paving $\mathcal{P}$ such that for any box p in $\mathcal{P}$, the set $\mathbb{S} \cap p$ is starry.

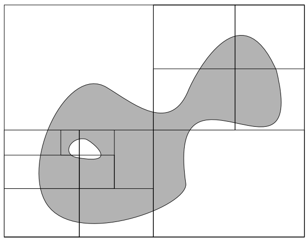

Figure 13: Example of paving $\mathcal{P}$ satisfying $\forall p \in \mathcal{P}, \mathbb{S} \cap p$ is starry

To any paving $\mathcal{P}$, it is possible to associate a graph $\mathcal{G}$ defined by

$$\mathcal{G} = \{(p, q) \in \mathcal{P} \times \mathcal{P}, \mathbb{S} \cap p \cap q \neq \emptyset\}.$$

For instance, the graph associated with the paving of Figure **??** is given on Figure **??**

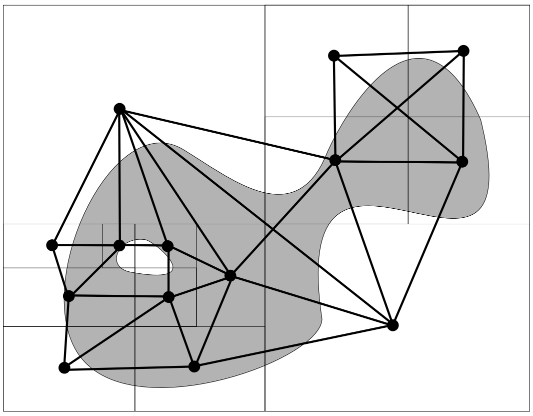

Figure 14: The graph $\mathcal{G}$ associated with the paving of the figure **??**

Proposition 4.1 *Let $\mathbb{S}$ be a subset of $\mathbb{R}^n$.*
Let $\mathcal{P}$ a paving of $X_0 \subset \mathbb{R}^n$.
If $\forall p \in \mathcal{P}$, the set $\mathbb{S} \cap p$ is starry and if the graph $\mathcal{G}$ associated with the paving $\mathcal{P}$ is connected then the set $\mathbb{S} \cap X_0$ is pathwise-connected.

Proof 4.1 If $\mathcal{G}$ is connected, therefore there exists a path from any node to any other node in the graph. Let n be the number of nodes, and $\mathcal{N} = (\alpha_i)_{i \in \{1,\ldots,n\}}$ be the nodes. Since $\mathcal{G}$ is connected, for all i in $\{1, \ldots, n-1\}$, there exists a path connecting α_i to α_{i+1}, i.e. there exists a finite sequence $\{\alpha_{i_1}, \alpha_{i_2}, .., \alpha_{i_k}\} \in \mathcal{N}^k$ such that $(\alpha_{i_1}, \alpha_{i_2}), (\alpha_{i_2}, \alpha_{i_3}), \ldots, (\alpha_{i_{k-1}}, \alpha_{i_k})$ are graph edges of $\mathcal{G}$ (with $\alpha_{i_1} = \alpha_i,\ \alpha_{i_k} = \alpha_{i+1}$). Let us note this path by $p(\alpha_i, \alpha_{i+1})$.
Let $path_1$ and $path_2$ be two pathes of $\mathcal{G}$.
If one of the endpoints of $path_1$ is one of the endpoints of $path_2$, then it is possible to create a new path from $path_1$ and $path_2$, noted $path_1 + path_2$, which is the concatenation of $path_1$ and $path_2$.
Let p_{all} be the path defined by this associative operation :

$$p_{all} = p(\alpha_1, \alpha_2) + p(\alpha_2, \alpha_3) + \ldots + p(\alpha_{n-1}, \alpha_n).$$

So p_{all} is a path of $\mathcal{G}$ which visits each node at least once. Let us note $(\beta_i)_{i\in\{1,\dots,m\}}$ the sequence of nodes visited by p_{all} with $\beta_1 = \alpha_1$ and $\beta_m = \alpha_n$.
Thus the sequence of boxes $(p_i)_{i\in\{1,\dots,m\}}$, where p_i is the box associated to the node β_i, satisfies :

$$\begin{cases} \forall i \in \{1,\dots,m\}, & p_i \cap \mathbb{S} \text{ is pathwise-connected (because } v^*_{p_i} \text{ is a star for } p_i \cap \mathbb{S}) \\ \forall i \in \{2,\dots,m\}, & \mathbb{S} \cap p_{i-1} \cap p_i \neq \emptyset. \end{cases}$$

To conclude, using the lemma : *Let $(A_i)_i$ be a denumerable family of pathwise-connected set*

$$\textit{if } \forall i \in I^*, A_{i-1} \cap A_i \neq \emptyset \textit{ then } \bigcup_{i\in I} A_i \textit{ is pathwise-connected}$$

we can say that

$$\bigcup_{i\in I} (p_i \cap \mathbb{S}) = \mathbb{S} \cap X_0 \text{ is pathwise-connected.}$$

5 Algorithm for proving that a set is pathwise-connected and examples

5.1 The main idea

This section presents a new algorithm called : `CIA` (pathwise-Connected using Interval Analysis). This algorithm tries to generate a paving which satisfies the hypothesis of proposition 4.1.
The main idea is to test a suggested paving $\mathcal{P}$ and, in the case where it does not satisfy the hypothesis, to improve this one by bisecting any boxes responsible for this failure.

5.2 Algorithms

To test a paving $\mathcal{P}$, we have to be able to prove that for a box p, $\mathbb{S} \cap p$ is starry or not, and to build its associated graph. This two tasks will be done by Alg. 2 and Alg. 3 respectively.

In `CIA` Alg. 1, $\mathcal{P}_*$, $\mathcal{P}_{out}$, $\mathcal{P}_\Delta$ are three pavings such that $\mathcal{P}_* \cup \mathcal{P}_{out} \cup \mathcal{P}_\Delta = \mathcal{P}$ is a paving of the initial box X_0 :

1. the *spangled paving* $\mathcal{P}_*$ is a paving containing only boxes p such that $\mathbb{S}\cap p$ is starry. Its associated graph interval is punctual. ($i.e.$, $\underline{g} = \overline{g}$, see Alg. 3).
2. the *outer* paving $\mathcal{P}_{out}$ is a paving containing only boxes p such that $\mathbb{S} \cap p$ is not starry.
3. the *uncertain* paving $\mathcal{P}_\Delta$, nothing is known about its boxes.

Alg. 1 `CIA` - pathwise-Connected using Interval Analysis

Input: $\mathbb{S}$ a subset of $\mathbb{R}^n$, X_0 a box of $\mathbb{R}^n$

1: $\mathcal{P}_* := \emptyset$
2: $\mathcal{P}_\Delta := \{X_0\}$
3: $\mathcal{P}_{out} := \emptyset$
4: **while** $\mathcal{P}_\Delta \neq \emptyset$ **do**
5:    Pull $\mathcal{P}_\Delta$ into the box p
6:    **if** $\mathbb{S} \cap p$ is not starry can be proved **Then** Push $\{p\}$ into $\mathcal{P}_{out}$, **Goto** Step 4.
7:    **if** $\mathbb{S} \cap p$ is starry can be proved and `Build_Interval_Graph`($\mathbb{S}, \mathcal{P}_* \cup \{p\}$) is punctual
8:             **Then** Push $\{p\}$ into $\mathcal{P}_*$, **Goto** Step 4.
9:    $Bisect(p)$ and Push the two resulting boxes into $\mathcal{P}_\Delta$
10: **end while**
11: $[\underline{g}, \overline{g}] :=$ `Build_Interval_Graph`($\mathbb{S}, \mathcal{P}_*$)
12: **if** $\underline{g}$ is connected **then**
13:    return "$\mathbb{S} \cap X_0$ is pathwise-connected"
14: **else**
15:    return "$\mathbb{S} \cap X_0$ is not pathwise-connected"
16: **end if**

To prove "$\mathbb{S} \cap p$ is starry", it suffices to check if one of the vertexes v_p of p is a star for $\mathbb{S} \cap p$. When $\mathbb{S}$ is defined by $f^{-1}(\mathbb{R}^-)$, this problem amounts the inequalities

$$\begin{cases} f(x) = 0 \\ Df(x).(x - v^*) \leq 0 \end{cases} . \tag{6}$$

has no solution in p. Then, an interval algorithm is used to check the inconsistency of the inequalities (6). The following algorithm called `Starry` shows how this verification can be implemented. [7])

Alg. 2 `Starry`(p, f)

Input: f a C^1 function from $\mathbb{R}^n$ to $\mathbb{R}$
Input: p a box of $\mathbb{R}^n$

1: **if** $f(p)$ can be proved to be inside $\mathbb{R}^{+*}$ **then**
2:    Return "$\mathbb{S} \cap p$ is empty thus it is not starry"
3: **else**
4:    **for all** vertex v_p of p **do**
5:       **if** the set $\{x \in p, f(x) = 0, Df(x).(x - v^*) \leq 0\}$ is inconsistent **then**
6:          Return "$\mathbb{S} \cap p$ is starry "
7:       **end if**
8:    **end for**
9:    Return "Failure"
10: **end if**

To build the associated graph of a paving $\mathcal{P}$, we have to check if for each pair (p_i, p_j) of the paving $\mathcal{P}$, $\mathbb{S} \cap p_i \cap p_j$ is empty or not. When we don't know if $\mathbb{S} \cap p_i \cap p_j$ is empty or not, We create a graph interval which contains the true graph. The following algorithm called `Build_Graph_Interval` shows how the graph building can be implemented :

Alg. 3 `Build_Graph_Interval`$(\mathbb{S}, \mathcal{P})$

Input: $\mathbb{S}$ a subset of $\mathbb{R}^n$, $\mathcal{P}$ a paving

Output: A graph interval $[\underline{g}, \overline{g}]$ associated to the paving $\mathcal{P}$.

1: $\overline{g} := \emptyset$
2: $\underline{g} := \emptyset$
3: **for all** (p_i,p_j) in $\mathcal{P} \times \mathcal{P}$ **do**
4:     **if** $\mathbb{S} \cap p_i \cap p_j = \emptyset$ can be proved **then**
5:         **next**
6:     **else**
7:         **if** For one of the vertexes v of p, $v \in \mathbb{S}$ **then**
8:             Add (p_i, p_j) to $\underline{g}$ and add (p_i, p_j) to $\overline{g}$
9:         **else**
10:             Add (p_i, p_j) to $\overline{g}$ *// i.e.*(p_i,p_j) *is an undetermined edge of* $[\underline{g}, \overline{g}]$
11:         **end if**
12:     **end if**
13: **end for**

When $\mathbb{S}$ is defined by inequalities, condition at step 4 is checked using interval arithmetic. With this tool, we can prove that $\mathbb{S} \cap p_i \cap p_j = \emptyset$.

5.3 Example

The `CIA` solver can be used to prove the pathwise-connectedness of a set defined by a C^1 function. Figure **??** shows the paving generated for

$$\mathbb{S} = \{(x, y) \in \mathbb{R}^2, f(x, y) \leq 0\}$$

with $f(x, y) = -e^{-4x^2} - e^{-(\frac{1}{2}x-2)^2} - e^{-(10x-12)^2} + \frac{1}{10} + y^2$.

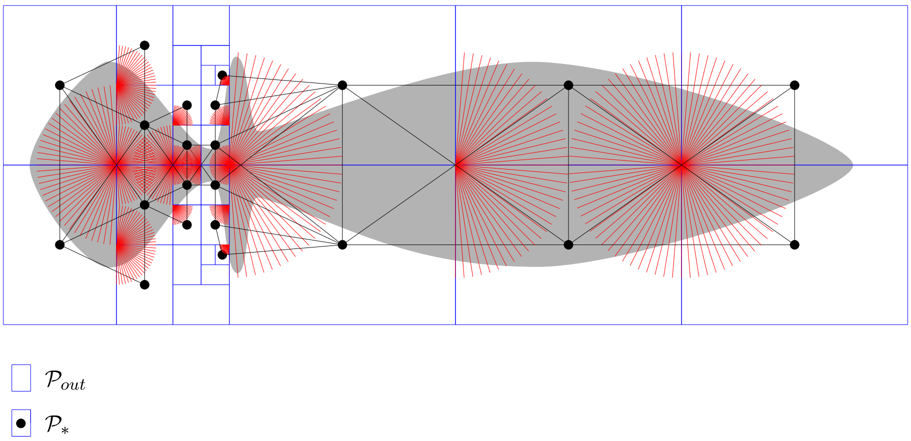

Figure 15: Example of paving generated by `CIA`

When the solver proves that a vertex of a box p is a star for $\mathbb{S} \cap p$, it uses the same representation that in Figure **??** to display it. (This solver can be downloaded from http://www.istia.univ-angers.fr/~delanoue/)

6 Conclusion

In this paper, an approach has been proposed to prove that a set $\mathbb{S}$ defined by inequalities is or not pathwise-connected. Combining tools from interval arithmetic and graph theory, an algorithm has been presented to build a graph which has some topological properties in common with $\mathbb{S}$. For instance, the number of pathwise-connected components of $\mathbb{S}$ is the same as its associated graph. The approach can thus be extended to count the number of pathwise-connected components of a subset of $\mathbb{R}^n$. One of the main limitation of the proposed approach is that the computing time increases exponentially with respect to the dimension of $\mathbb{S}$.

As an extension of this work, is the problem of the computation of a triangulation homeomorphic to $\mathbb{S}$. Roughly speaking, a triangulation is a nonoverlapping union of simplexes. This would make it possible to get some topological properties of the set of interest such as its homology groups. We hope that this problem could be solved by combining the tools presented in this paper with algorithms arising out from *Smith normal form* (see [10]).

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