A Verification Method for Solutions of Linear Programming Problems

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The Linear Programming Problem (LP)

Primal Form:

$$\begin{aligned} \text{Minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{[P]}$$

(m equalities, n variables)

Dual Form:

$$\begin{aligned} \text{Maximize} \quad & b^\top y \\ \text{subject to} \quad & A^\top y + z = c \\ & z \geq 0 \end{aligned} \qquad \text{[D]}$$

Interior-Point Methods (IPM)

Idea: Solve LP via sequence of smooth nonlinear unconstrained optimization problems

Fundamental Components of IPM

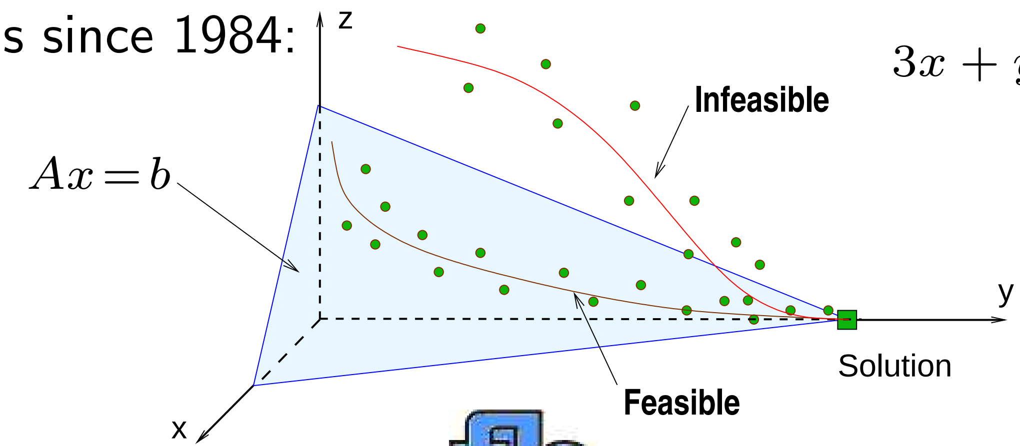
- *Logarithmic barrier method* to deal with a constrained optimization problem with inequality constraints
- *Lagrangian function* to transform a constrained optimization problem with equality constraints into an equivalent unconstrained problem
- *Newton's method* to solve unconstrained minimization problem by finding a zero of the gradient

Early Methods (1950s – 1960s)

- $F(x) = -\log x\,;\ x > 0$ (Frisch 1955)
- SUMT approach (Fiacco & McCormick 1968)

New Methods (1984 –)

- new polynomial-time Method for LP (Karmarkar 1984)
- equivalence with early IPM (Gill, Murray, Saunders, Tomlin & Wright, 1986)
- many variations since 1984:



$$3x + y + 2z = 1$$
$$x, y, z \geq 0$$

Motivation and Goals

1. Primal-dual logarithmic barrier methods (Kojima, Mizuno & Yoshise, 1990)
   - currently the most efficient computational methods
   - simultaneous solution for [P] and [D] with duality gap $c^\top x - b^\top y \leq \epsilon$
   - If the starting point is infeasible, then the computed solution may be infeasible
2. Reliability of conventional computer program?
   - floating point arithmetic: roundoff error, cancellation, . . .
   - quality of approximate solution?
   - results occasionally qualitatively wrong

Verified Computation

- verification of existence and uniqueness of solution in an interval
- guaranteed enclosure with narrow bounds for the optimal objective function value

Primal-Dual Logarithmic Barrier Methods

Barrier Problems:

Minimize	$c^\top x - \mu \sum_{i=1}^{n} \ln x_i$	[P$_\mu$]
subject to	$Ax = b$	

Maximize	$b^\top y - \mu \sum_{i=1}^{n} \ln z_i$	[D$_\mu$]
subject to	$A^\top y + z = c$	

Lagrangian Functions:

$$L(x, y, \mu) = c^\top x - \mu \sum_{j=1}^{n} \ln x_j - y^\top (Ax - b)$$

$$L(x, y, z, \mu) = b^\top y + \mu \sum_{j=1}^{n} \ln z_j - x^\top (A^\top y + z - c)$$

Optimality Conditions

$$\begin{aligned} Ax &= b\,, \\ A^\top y + z &= c\,, \\ XZe &= \mu e\,, \end{aligned} \qquad \text{[KKT]}$$

where $X = \text{diag}(x_1, x_2, \ldots x_n)\,,\; Z = \text{diag}(z_1, z_2, \ldots z_n)\,,\; e = (1, 1, \ldots, 1)^\top$

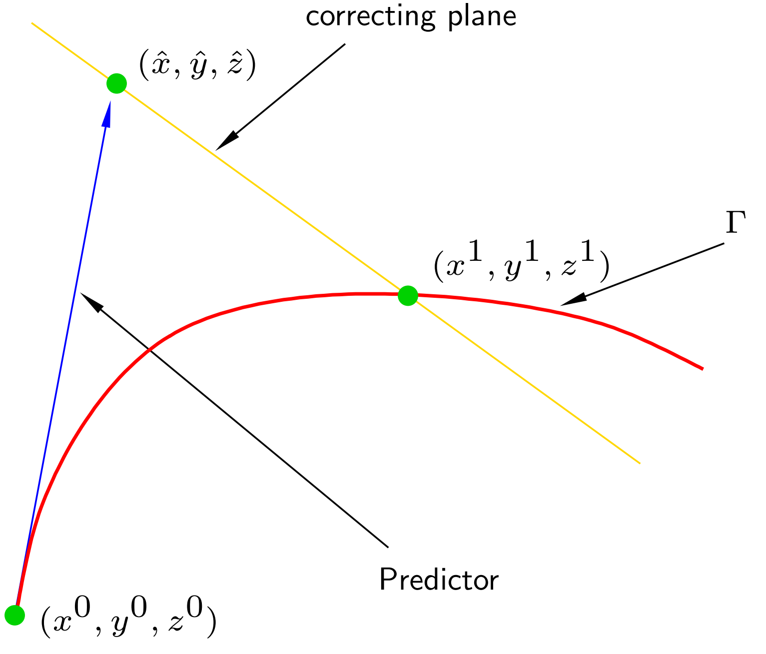
Iterative Solution

$$\begin{aligned} \Delta y &= -(AXZ^{-1}A^\top)^{-1}AZ^{-1}v(\mu)\,, \\ \Delta z &= -A^\top \Delta y\,, \\ \Delta x &= Z^{-1}v(\mu) - XZ^{-1}\Delta z\,, \end{aligned}$$

where $v(\mu) = \mu e - XZe$

Mehrotra's Predictor-Corrector Method

- The goal is to follow the central path $(x(\mu), y(\mu), z(\mu))$
- Predictor-corrector method proceeds as follows:



A Verification Method

LP as root finding problem

$$F(x,y,z) = \begin{pmatrix} Ax - b \\ A^\top y + z - c \\ XZe - \mu e \end{pmatrix} = 0$$

Verification Problem

$$u = (x,y,z),$$

$$\boldsymbol{u} = \{(u_1, u_2, \ldots, u_{2n+m})^\top \in I\!R^{2n+m} \mid \underline{u}_i \le u_i \le \overline{u}_i,\ 1 \le i \le 2n+m\}$$

`Given:` $F : I\!I\!R^{2n+m} \to I\!I\!R^{2n+m}$

`Verify:` There exists a unique $u^* \in \boldsymbol{u}$ such that $F(u^*) = 0$

Techniques for Verification

- *Interval Arithmetic*: $\circ \in \{+, -, \cdot, /\}$, $\boldsymbol{u}, \boldsymbol{v} \in I\mathbb{R}$

$$\begin{aligned} \boldsymbol{u} \circ \boldsymbol{v} &= \{u \circ v \,|\, u \in \boldsymbol{u},\, v \in \boldsymbol{v}\} = [\underline{w}, \overline{w}] \subseteq \boldsymbol{u} \diamond\!\!\!\circ\, \boldsymbol{v} = [\triangledown \underline{w}, \triangle \overline{w}] \\ W_F(\boldsymbol{u}) &= \{F(u) \mid u \in \boldsymbol{u}\} \subseteq F(\boldsymbol{u}) \subseteq F_\diamond(\boldsymbol{u}) \end{aligned}$$

- *Inclusion property*:

$$\begin{aligned} \boldsymbol{u} \subseteq \boldsymbol{u}', \boldsymbol{v} \subseteq \boldsymbol{v}' &\Rightarrow \boldsymbol{u} \circ \boldsymbol{v} \subseteq \boldsymbol{u}' \circ \boldsymbol{v}' \\ \boldsymbol{u} \subseteq \boldsymbol{u}' &\Rightarrow F(\boldsymbol{u}) \subseteq F(\boldsymbol{u}') \end{aligned}$$

- *Mean value form*: $u^* = \text{mid}(\boldsymbol{u})$

$$F(u) \in F(u^*) + J_F(\boldsymbol{u})(\boldsymbol{u} - u^*), \ \text{ for } \ u, u^* \in \boldsymbol{u}$$

- *Brouwer's fixed point theorem*: Let $\boldsymbol{u}$ be a closed, bounded and convex set in $\mathbb{R}^n$ and let F be a continuous mapping such that $F : \boldsymbol{u} \to \boldsymbol{u}$, then F has a fixed point in $\boldsymbol{u}$.

$$\boldsymbol{u} \in I\mathbb{R}^n : F(\boldsymbol{u}) \subseteq F_\diamond(\boldsymbol{u}) \subseteq \boldsymbol{u} \Rightarrow \exists\, \boldsymbol{u}^* \in \boldsymbol{u} \mid \boldsymbol{u}^* = F(\boldsymbol{u}^*)$$

A verification method can be defined by an iteration of the form:

$$\tilde{u}^k = \text{mid}(\boldsymbol{u}^k)$$

$$R = \text{mid}\big(J_F(\boldsymbol{u}^k)\big)^{-1}$$

$$K(\boldsymbol{u}, \tilde{u}^{k+1}) = \tilde{u} - R \cdot F(\tilde{u}^{k+1}) + (I - R \cdot J_F(\boldsymbol{u}))(\boldsymbol{u} - \tilde{u}^{k+1})$$

Some Notes

1. Under normal circumstances, every zero of F in $\boldsymbol{u}$ can be enclosed.
2. If there is no zero of F in $\boldsymbol{u}$, the algorithm is generally able to prove this fact automatically in a finite number of iterations
3. Proof of existence or nonexistence of a zero of F in a given interval $\boldsymbol{u}$ occurs automatically:
   - If $K(\boldsymbol{u}, \tilde{u}) \overset{\circ}{\subset} \boldsymbol{u}$, then there exists a zero of F in $\boldsymbol{u}$
   - If $K(\boldsymbol{u}, \tilde{u}) \cap \boldsymbol{u} = \emptyset$, then there is no zero of F in $\boldsymbol{u}$

Verified Linear Programming Algorithm

(1) Let a primal problem [P] and its dual [D] be given

(2) Compute an approximate solution $(\tilde{x}, \tilde{y}, \tilde{z})$ for [P] and [D] by performing a primal-dual logarithmic barrier algorithm

(3) Construct a small box $\boldsymbol{u}$ about the approximate solution $\tilde{u} = (\tilde{x}, \tilde{y}, \tilde{z})$ by using an epsilon–inflation algorithm

(4) Perform a verification step:

(a) Prove the existence and uniqueness of a zero of F

(b) Enclose the optimal objective function value by:

$$c^\top \boldsymbol{x} \cup b^\top \boldsymbol{y}$$

Numerical Results

Example 1

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 100 & 18 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 50 \\ 200 \\ 5000 \end{pmatrix} \quad \text{and} \quad c^\top = (-50,\ -9,\ 0,\ 0,\ 0)$$

Iteration count	Primal objective	Dual objective	Primal infeasibility	Dual infeasibility	Relative gap
0	$-0.59000000E+02$	$-0.52500000E+04$	$0.49E+04$	$0.51E+02$	$0.52E+04$
1	$-0.24989469E+03$	$-0.50203503E+04$	$0.45E+04$	$0.47E+02$	$0.48E+04$
2	$-0.32441434E+03$	$-0.49380485E+04$	$0.43E+04$	$0.45E+02$	$0.46E+04$
⋮	⋮	⋮	⋮	⋮	⋮
8	$-0.24915636E+04$	$-0.25090225E+04$	$0.17E+02$	$0.17E+00$	$0.17E+02$
⋮	⋮	⋮	⋮	⋮	⋮
12	$-0.25000000E+04$	$-0.25000000E+04$	$0.32E-06$	$0.34E-08$	$0.34E-06$
13	$-0.25000000E+04$	$-0.25000000E+04$	$0.45E-11$	$0.50E-13$	$0.48E-11$

Result of verification: Normal end, the problem has an optimal solution, an inclusion of the optimal value has been computed.
$[-2.500000000000019E3, -2.499999999999981E3]$

Example 2

$$A = \begin{pmatrix} -1 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 10^{-6} \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Iteration count	Primal objective	Dual objective	Primal infeasibility	Dual infeasibility	Relative gap
0	$0.30000000E+03$	$-0.10000000E-04$	$0.20E+03$	$0.60E+01$	$0.30E+03$
1	$0.19091192E+03$	$-0.44436010E-09$	$0.13E+03$	$0.38E+01$	$0.19E+03$
2	$0.25015137E+02$	$-0.13761119E-04$	$0.18E+02$	$0.53E+00$	$0.25E+02$
⋮	⋮	⋮	⋮	⋮	⋮
8	$0.27721815E-04$	$-0.78569105E-05$	$0.26E-04$	$0.79E-06$	$0.36E-04$
9	$0.25064585E-05$	$-0.70012575E-05$	$0.74E-05$	$0.22E-06$	$0.95E-05$
⋮	⋮	⋮	⋮	⋮	⋮
14	$0.16550663E-07$	$-0.82880998E-08$	$0.22E-07$	$0.66E-09$	$0.25E-07$
15	$0.62150694E-11$	$-0.41463218E-12$	$0.98E-11$	$0.30E-12$	$0.66E-11$

Result of verification: The interval $\boldsymbol{u}$ contains no solution of $F(u) = 0$.

blend (NETLIB-test-problem with 75 equalities and 83 variables)

Iteration count	Primal objective	Dual objective	Primal infeasibility	Dual infeasibility	Relative gap
0	$-0.16500200E+02$	$-0.11191000E+03$	$0.25E+03$	$0.10E+03$	$0.95E+02$
1	$-0.40473968E+02$	$-0.12064083E+03$	$0.21E+03$	$0.87E+02$	$0.80E+02$
2	$-0.33415294E+02$	$-0.11146155E+03$	$0.21E+03$	$0.85E+02$	$0.78E+02$
⋮	⋮	⋮	⋮	⋮	⋮
10	$-0.29243417E+02$	$-0.31416758E+02$	$0.58E+01$	$0.24E+01$	$0.22E+01$
11	$-0.29952535E+02$	$-0.30708204E+02$	$0.20E+01$	$0.84E+00$	$0.76E+00$
⋮	⋮	⋮	⋮	⋮	⋮
24	$-0.30812150E+02$	$-0.30812150E+02$	$0.11E-07$	$0.44E-08$	$0.39E-08$
25	$-0.30812150E+02$	$-0.30812150E+02$	$0.12E-10$	$0.56E-11$	$0.20E-12$

Result of verification: Normal end, the problem has an optimal solution,
an inclusion of the optimal value has been computed.
$[-3.0812149845852329E1, -3.0812149845852136E1]$

Conclusions

- Our results indicate that the presented algorithm is robust.
- A verification is obtained for all test problems.
- The verification step is computationally the most expensive part of the algorithm.
- The best choice of a default starting point is still very much an open question.
- IPMs have been extended to more general classes of problems, but verified algorithms have not yet appeared.