

EGYPTIAN FRACTIONS

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1. Introduction

The Egyptian fraction expansion of a rational $0 < a/b < 1$ refers to a solution in integers $0 < x_1 < \dots < x_n$ of

$$\frac{a}{b} = \frac{1}{x_1} + \dots + \frac{1}{x_n},$$

with n minimal. Although this expansion is not well defined, for example $\frac{2}{105} = \frac{1}{57} + \frac{1}{665} = \frac{1}{65} + \frac{1}{273}$, its length is. P. Erdős [1] has proved that $n \ll \frac{\log b}{\log \log b}$. In this paper we improve the bound to $n \ll \sqrt{\log b}$.

2. Results

The improved upper bound follows (employing the same argument used by Erdős [1]) from

THEOREM 1. *There exists an increasing sequence N_k of positive integers such that any integer $1 < m < N_k$ is the sum of not more than $O(\sqrt{\log N_{k-1}})$ distinct divisors of N_k .*

Assuming Theorem 1, let $0 < a/b < 1$ and choose integers k and l such that

$$N_{k-1} < b \leq N_k \quad \text{and} \quad \frac{l}{N_k} \leq \frac{a}{b} < \frac{l+1}{N_k}.$$

It follows that

$$aN_k - bl < b \leq N_k \quad \text{and} \quad l < N_k.$$

Theorem 1 gives the representations

$$aN_k - bl = d_1 + \dots + d_r \quad \text{and} \quad l = d'_1 + \dots + d'_s,$$

where d_i and d'_j are divisors of N_k . Now define integers u_i and v'_j by

$$u_i = \frac{N_k b}{d_i}, \quad v'_j = \frac{N_k}{d'_j}.$$

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Since $d_r \leq aN_k - bl < b$, it follows that $v_1 \leq N_k < N_k b/d_r = u_r$. This proves that

$$v_s < \dots < v_1 < u_r < \dots < u_1,$$

provided $d_1 < \dots < d_r$ and $d'_1 < \dots < d'_s$. Now observe that

$$\frac{1}{v_1} + \dots + \frac{1}{v_s} + \frac{1}{u_1} + \dots + \frac{1}{u_r} = \frac{1}{N_k} \left(l + \frac{aN_k - bl}{b} \right) = \frac{a}{b},$$

and $r+s = O(\sqrt{\log N_{k-1}}) = O(\sqrt{\log b})$.

Before proving Theorem 1 we establish some notation. Let $H(N)$ be the smallest integer such that every integer m , $0 < m < N$, is the sum of not more than $H(N)$ distinct divisors of N . For real x , $1 < x < N$, let $d^-(x)$ and $d^+(x)$ be consecutive divisors of N such that $d^-(x) < x \leq d^+(x)$.

LEMMA 1. Suppose that N_0, N are integers such that $N_0 \mid N$, and $H(N_0)$ exists. Let $N_0 = d_1 < \dots < d_l = d^+(\sqrt{N})$ be a sequence of divisors of N such that

$$(*) \quad d_j \left(\max_{d_{j-1} < i \leq d_j} \left(1 - \frac{d^-(i)}{d^+(i)} \right) \right) \leq d_{j-1}$$

for $1 < j \leq l$. Then $H(N) \leq H(N_0) + 2l$.

LEMMA 2. There exists a sequence of positive integers of the form $N_k = 4^{\alpha k^2} \prod_{i=2}^k p_i^2$, where $p_2 < p_3 < \dots$ are odd primes, such that

- (i) $\max_{\sqrt{N_{k-1}} < i \leq \sqrt{N_k}} (\log d^+(i) - \log d^-(i)) \ll (2/3)^k$
- (ii) $\log p_i \asymp i$
- (iii) α may be any sufficiently large integer.

Proof of Theorem 1. The required sequence is $1, 2, N_2, N_3, \dots$ where the N_k are the integers from Lemma 2. First note that $H(N_k)$ exists. This follows from the observation that for sufficiently large α ,

$$\log \sqrt{N_k} = \alpha k^2 \log 2 + \sum_{i=2}^k \log p_i < \alpha k^2 \log 4.$$

Therefore

$$\max_{1 < i \leq \sqrt{N_k}} \left(1 - \frac{d^-(i)}{d^+(i)} \right) \leq \frac{1}{2},$$

and Lemma 1 can be applied using the sequence $2^1, 2^2, 2^3, \dots$. The proof of Theorem 1 is completed by constructing, for $k > K$, a sequence (of length $O(\sqrt{\log N_{k-1}})$) of divisors of N_k to which we may apply Lemma 1 with $N_0 = N_k$.

Since $-\log x \geq 1 - x$, we have (Lemma 2)

$$c(2/3)^k \geq \log d^+(i) - \log d^-(i) \geq 1 - \frac{d^-(i)}{d^+(i)}, \quad (1)$$

for $\sqrt{N_{k-1}} < i \leq \sqrt{N_k}$ and some constant c . Therefore

$$d^+(i) < i + d^+(i) \left(1 - \frac{d^-(i)}{d^+(i)}\right) \leq i + d^+(i)r_k, \quad (2)$$

where $r_k = c(2/3)^k$. We assume K is sufficiently large that $r_k < \frac{1}{2}$. Recursive use of (2) produces

$$d^+(i) < i \sum_{j=0}^{\infty} r_k^j = \frac{i}{1-r_k}. \quad (3)$$

Define a nonincreasing sequence of divisors (of N_k) for $j = 1, \dots, J_k + 1$ by

$$d_1(k) = \sqrt{N_k}, \quad d_{j+1}(k) = d^+(d_j(k)r_k).$$

Suppose that J_k can be defined by the condition

$$d_{J_k+1}(k) \leq \sqrt{N_{k-1}} < d_{J_k}(k). \quad (4)$$

Then the inequality $d_j(k)r_k \leq d_{j+1}(k)$ and (1) imply that condition (*) of Lemma 1 holds for the set of divisors (when reindexed according to increasing size) $\{d_1(k), \dots, d_{J_k}(k), \sqrt{N_{k-1}}\}$. Condition (4) can be satisfied if for some J , $d_{J+1}(k) \leq \sqrt{N_{k-1}}$. By the definition of $d_j(k)$ and inequality (3), it suffices that

$$\left(\frac{r_k}{1-r_k}\right)^J \leq \sqrt{\frac{N_{k-1}}{N_k}}.$$

This is equivalent to

$$J \geq \frac{\alpha(2k-1) \log 2 + \log p_k}{\log(r_k^{-1} - 1)},$$

and since $\log p_k \asymp k$, the right hand side is $O(1)$. Therefore we may assume that $J_k = O(1)$ as $k \rightarrow \infty$.

The desired sequence of divisors (of N_k , to which we can apply Lemma 1 with $N_0 = N_k$) is given by those elements of the set

$$\{d_j(i) : K < i \leq k, 1 \leq j \leq J_i\} \cup \{N_k\}$$

which are not smaller than N_k . By construction, condition (*) of Lemma 1 holds, and the length of the sequence is bounded by $1 + \sum_{K < i \leq k} J_i = O(k)$. Therefore Lemma 1 gives

$$H(N_k) \leq H(N_K) + O(k) = O(\sqrt{\log N_{k-1}}).$$

Proof of Lemma 1. First suppose that $N_0 < m < d^+(\sqrt{N})$. Note that if m divides N then we are done. Hence m lies in an interval of the form (d_{j-1}, d_j) for some $j \leq l$, and

$$m - d^-(m) < d^+(m) \left(1 - \frac{d^-(m)}{d^+(m)} \right).$$

It follows from condition (*) that $m - d^-(m) < d_{j-1}$. Therefore $m - d^-(m)$ lies in an interval of the form $[d_{i-1}, d_i)$ for some $i < j$, or else $m - d^-(m) < N_0$. It follows that m is the sum of not more than $H(N_0) + l$ distinct divisors of N .

Now consider the case $d^+(\sqrt{N}) < m < N$. Let $d_0 = 1$. As before, m lies in an interval of the form $(Nd_j^{-1}, Nd_{j-1}^{-1})$ for some $j > 0$, and

$$m - d^-(m) < \frac{N}{d_{j-1}} \left(\max_{N/d_j < m < N/d_{j-1}} \left(1 - \frac{d^-(m)}{d^+(m)} \right) \right).$$

The maximum above is

$$\max_{\substack{d|N \\ d_{j-1} \leq N/d < d_j}} \left(1 - \frac{N/d}{N/d^-(d)} \right) = \max_{d_{j-1} < m \leq d_j} \left(1 - \frac{d^-(m)}{d^+(m)} \right),$$

and it follows that $m - d^-(m) < Nd_j^{-1}$. Therefore $m - d^-(m)$ lies in an interval of the form $[Nd_i^{-1}, Nd_{i-1}^{-1})$ for some $i > j$, or else $m - d^-(m) < d^+(\sqrt{N})$. Hence m is the sum of not more than $H(N_0) + 2l$ distinct divisors of N .

We refer the reader to [2] for the proof of Lemma 2 since it is the content of Lemma 3 of that paper.

References

1. P. ERDŐS, 'The solution in whole number of the equation $1/x_1 + \dots + 1/x_n = a/b$ ', *Mat. Lapok*, 1 (1950), 192-210.
2. M. VOSE, 'Integers with consecutive divisors in small ratio', *J. Number Theory* (to appear).

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